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Comparative Analysis of the Properties of the Grötzsch, Thomassen, and Herschel Graphs: A Spectral and Algebraic Approach

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Abstract

This paper analyzes and compares the fundamental properties of three well-known graphs: the Grötzsch graph, the Thomassen graph, and the Herschel graph. From the class of Mycielski graphs, the Grötzsch graph is chosen as a representative with the smallest chromatic number of 4 and no triangles. The Grötzsch theorem is discussed in detail, as well as recent results connecting this theorem with the corresponding graph. Next, the spectral characteristics of the Thomassen and Herschel graphs are analyzed, with special emphasis on their algebraic structure, which is particularly significant for the application in studies involving large graphs. The research includes the calculation of the eigenvalues and eigenvectors of the Laplacian matrices of these graphs, providing insight into their spectral properties. Modern computational tools are utilized for the visualization and analysis of large graphs, such as the Thomassen graph with 42 vertices. The paper presents the basics of graph theory through isomorphic representations of the three graphs, as well as the theoretical framework necessary for understanding their intrinsic properties, which is the ultimate goal of this research.

Keywords: Graph theory, chromatic number, Grötzsch graph, Thomassen graph, Herschel graph, graph spectrum

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1. Introduction

Graph theory is a branch of mathematics that deals with the study of graphs, which are structures composed of nodes (also called vertices) and the branches (or edges) that connect them. Graphs are used to model many real-world problems, such as computer networks, social networks, transportation systems, and many others. Thanks to its applicability, graph theory has become an indispensable tool in various scientific disciplines, including mathematics, computer science, physics, biology, and the social sciences. Within this theory, numerous concepts and algorithms have been developed that enable analysis of the structure and dynamics of networks, as well as solving problems of optimization, search, and analysis of large data sets. Graph theory is a powerful tool for solving problems in various fields and continues to be developed to provide solutions for complex real situations. Basic concepts in graph theory include nodes, which represent entities, and branches, which represent relationships or connections between those entities. There are several types of graphs, including directed and undirected graphs, connected and disconnected graphs, and weighted graphs where branches have associated values or weights. Each of these types has specific properties that make it suitable for different types of analysis.

Graph theory has wide applications in the real world. For example, in the field of information technology, it is used to analyze computer networks, optimize data traffic, and find the shortest paths. In the social sciences, graphs are used to analyze social networks and study interactions between individuals or groups. In biology, graphs are used to model genetic and protein networks, allowing a better understanding of complex biological processes. Within graph theory, numerous search, analysis and optimization algorithms have been developed, such as algorithms for finding the shortest path, cluster identification, cycle detection and many others. Due to its universality and flexibility, graph theory represents a powerful mathematical framework for solving a wide range of problems in different scientific disciplines.

2. The Mycielski and Grötzsch Graphs

The Mycielski and Grötzsch graphs are specific types of graphs in graph theory that are used to study graph coloring, especially the problem of the minimum number of colors needed to color graphs without certain structures, such as triangles.

2.1. **The Mycielski graphs**

Mycielski graphs MnM_nMn are a special class of graphs that are constructed to be three-color uncolorable, but have no triangles (ie, no cliques of size three). They are interesting because they represent counterexamples in graph theory: although they are tricolor uncolorable, their chromatic number (the minimum number of colors needed to color nodes so that no two adjacent nodes have the same color) is greater than the number ω\omegaω (the largest number of mutually adjacent nodes).

2.1.1. Construction of Mycielski graph:

1. We start with a simple graph without triangles GGG with chromatic number nnn.
2. For each node uuu in the graph GGG, a new node u′u'u′ is added, which is connected to all the neighbors of the node uuu in GGG.
3. A new node www is added which is connected to all added nodes u′u'u′.

In this way, every Mycielski graph MnM_nMn has no triangles, but has chromatic number n+1n+1n+1.

2.2. The Grötzsch Graph

A Grötzsch graph is an example of a **4-color disjoint plan** (a graph without triangles that can be drawn on a plane without intersecting branches) that is tricolor. This means that all the nodes of the graph can be colored with three colors, so that no two adjacent nodes have the same color, without containing any triangles.

2.2.1. Properties of the Grötzsch graph:

1. **Chromatic number:** Grötzsch graph is tricolor, which means its chromatic number is 3.
2. **Triangle-free graph:** Grötzsch graph has no triangles, making it a **clique-free graph of size 3**.
3. **Planar graph:** Can be drawn on a plane so that no two branches intersect.

Grötzsch's graph is one of the first examples to show that graphs without triangles can have a chromatic number of three, thereby disproving the assumption that all graphs without triangles are two-color (which is only true for graphs that are trees). Mycielski and Grötzsch graphs play an important role in graph theory, especially in the study of graph coloring problems. They provide insight into the complexity of triangle-free graphs and help understand the bounds and colorability of such structures.

In graph theory, the class of Mycielski graphs generated by some arbitrary initial graph is known. The Polish mathematician Jan Mycielski, in 1955, using the method that bears his name, proved that there are graphs that do not contain triangles and that have arbitrarily large chromatic numbers. That method (more precisely, the construction) generates Mycielski's classes.

Denote by G an arbitrary given graph, whose nodes are $v_1, v_2, ..., v_n$. We define the graph $m(G)$ of the graph *G*, as a graph whose subgraph is isomorphic to the graph *G* and which has *n+1* more nodes than *G*: let these additional nodes be $u_1, u_2, ..., u_n$, which correspond to the nodes $v_1, v_2, ..., v_n$, and we have another additonal node *w*. Each node u_i is connected to *w* via a branch (so these nodes form a star subgraph). Additionally, for each branch $v_i v_j$ of the graph *G*, the graph $m(G)$ also contains $u_i v_j$ and $v_i u_j$. The graph $m(G)$ is called the Mycielski graph for the graph *G*.

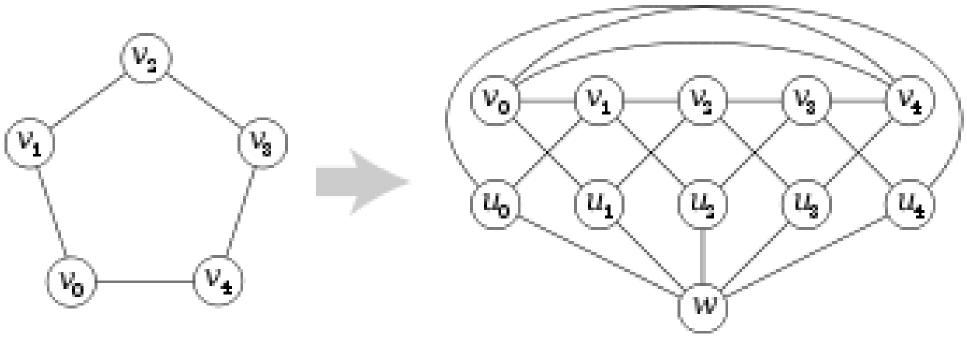

Figure 1: Example of obtaining a graph $m(G)$ from the initial graph G

By construction, if *G* has *n* nodes and *m* branches, $m(G)$ has *2n+1* nodes and *3m+n* branches. An example of Mycielski's graph construction is given in Figure 1.

By successively applying Mycielski's construction, for an initial graph consisting of only one node, we obtain a whole class - a series of graphs: $M_i = m(M_{i-1})$. The first few graphs of that class (basic Mycielski classes) are shown in Figure 2.

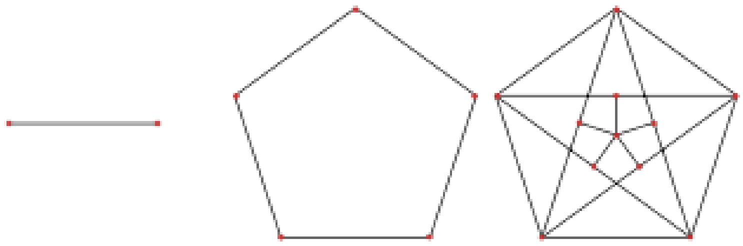

Figure 2: The first three Micielski graphs from the basic class

All graphs of the considered class, by construction, do not contain triangles in them, and each subsequent one in the sequence has a chromatic number one greater than its predecessor. However, it can be shown that the following, stronger statement is also valid:

(1) Every Mycielski graph of order *k*, is a nontriangular graph of chromatic number *k* with the smallest number of nodes.

Furthermore, Jan Mycielski himself and mathematicians after him proved some other statements, the most significant of which are the following two:

(2) If the graph *G* contains a Hamiltonian cycle, then $\mu(G)$ also contains it. (Fischer, McKenna, Boyer, 1998.);

(3) If a graph *G* is factor-critical, then $\mu(G)$ is also factor-critical (Došlić, 2005.).

Using the first statement, we obtain that the smallest non-triangular graph of the chromatic number 4 is the fourth member of the basic Mycielski family. That graph is better known as Grötzsch's graph (Herbert Grötzsch), whose two isomorphic representations and one of its Hamilton cycles are shown in Figure 3, and to which we will now pay a little more attention.

The complete group of automorphisms of the Grötzsch graph is isomorphic to the dihedral group D_5 of order 10. That group represents the group of symmetries of a regular pentagon with rotations and translations. We determine whether this graph may contain an Euler path by the well-known theorem:

Theorem 1: *A connected multigraph G with at least one branch has an Euler path if and only if it contains 0 or 2 nodes of odd degree..*

According to the above theorem, we easily get that Grötzsch's graph does not contain an Euler path. The Hamiltonian contour, however, contains it (effectively shown in Fig. 3b): the existence of 20 different Hamiltonian contours in this graph, as well as 980 trajectories, was determined.

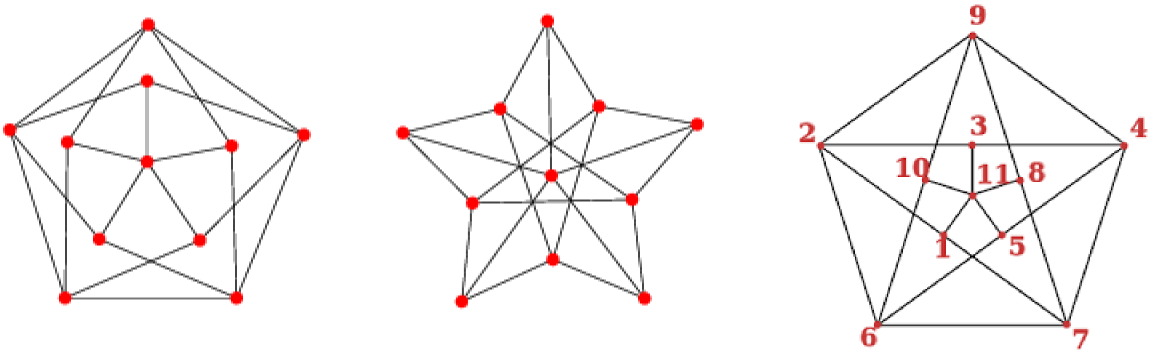

Figure 3: a) Grötzsch's graph – two isomorphic representations; b) Hamilton's cycle

Now let's analyze the chromatic polynomial of our graph. By standard procedures, now using the help of mathematical software packages, we get the characteristic polynomial:

$$P_G(x) = (x-1)^5(x^2 - x - 10)(x^2 + 3x + 1)^2$$

whose roots are: 1 (5 times), $\frac{1 \pm \sqrt{41}}{2}$ i $\frac{-3 \pm \sqrt{5}}{2}$. Obviously the whole spectrum is real. We can use the obtained spectrum to determine the regularity of the graph: we use the following theorem of the spectral theory of graphs, which is most often more useful in its reverse direction:

Theorem 2: *Let* $\lambda_1 (= r), \lambda_2, \lambda_3, \ldots \lambda_n$ *be the members of the spectrum of the graph G, and let r be its index, i.e. the largest characteristic value from the. The the graph G is regular, if and only if it holds that*

$$\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = r$$

In the case of Grötzsch's graph, calculating the corresponding sum, we get a discrepancy between the left and right sides of the equality, from which it follows that Grötzsch's graph is not regular. In the reverse direction of the theorem, we can draw a conclusion about the sum of the inverse elements of the spectrum of a graph if we know the fact whether it is regular or not.

Let us now recall the definition of the chromatic polynomial and the chromatic number of a graph.

Definition 1: *The chromatic polynomial P(G,L) of the graph G, for each positive l, has a value equal to the number of different regular l-colorings of the graph G. The chromatic number represents the minimum number of different colors required for the regular coloring of the nodes of the graph G.*

Now, using the *Mathematica* package, we quickly obtain the chromatic polynomial of the Grötzsch graph:

$$P(G,\lambda) = (\lambda-3)(\lambda-2)(\lambda-1)\lambda(\lambda^7 - 14\lambda^6 + 95\lambda^5 -$$
$$400\lambda^4 + 1115\lambda^3 - 2033\lambda^2 + 2217\lambda - 1100)$$

The first positive integer value for which the above polynomial has a value different from zero is four, which confirms and states from the beginning of this paper that Grötzsch's graph is 4-colorable.

The existence of Grötzsch's graph also plays an important role in the proof of the theorem of the same name. More precisely, it demonstrates that the graph planarity assumption in Grötzsch's theorem (which asserts that every planar nontriangular graph is 3-colorable, proved in 1959) is necessary: Grötzsch 's graph is not planar (observe its image).

We also believe that in the end it would not be out of place to mention (without proof) the following lemma that connects two graphs and their tensor product (we will assume that this product of graphs is known to readers) with their chromatic numbers:

Lemma 1 (Stefan Hedetniemi): *The chromatic number of the tensor product of graphs G and H, $G' \ H$, is equal to the minimum of the chromatic numbers of graphs G and H:*

$$c(G' \ H) = \min[c(G), c(H)]$$

3. The Thomassen Graph

The Thomassen graph is a specific graph in graph theory that is significant in the context of graph coloring. It was developed by the Danish mathematician Carsten Thomassen. This graph is used as a counterexample or illustration in various graph theory problems, especially in relation to the chromatic number and planar graphs.

3.1. Properties of the Thomassen graph

1. Planarity: The Thomassen graph is a **planar graph**, which means that it can be drawn on a plane without cutting branches. This is a feature that places it in a special class of graphs suitable for the study of plan and coloring.
2. No triangles: Thomasen's graph has no cycles of length three, i.e. does not contain triangles (size 3 cliques). This property makes it a **triangle-free graph**, which makes it suitable for exploring the minimum colors needed for coloring.
3. Chromatic number: Although it is triangle-free, the Thomassen graph can have a high chromatic number. For example, the **triangle-free Thomassen 5-chromatic planar graph** is known, which means that it needs at least five colors to be colored, even though it does not have any triangles. This is a significant counterpoint because it shows that even planar graphs without triangles need not be tricolor.

4. Construction: There are several different graphs called Thomassen graphs, and their construction often includes specific procedures that ensure the absence of triangles and the retention of a certain planar structure..

3.2. Applications and Significance

The Thomassen graph is important in graph theory because it shows extreme examples for certain chromatic properties of planar graphs. It is also used as a countermeasure in proving or disproving conjectures related to graph coloring, especially in situations where graphs with high chromatic number are studied in the absence of small cycles (such as triangles). Thomassen graphs serve as a useful tool for researchers in understanding the behavior of graphs in extreme cases and contribute to the development of graph coloring theory. Similar to Figure 2, Figure 4 shows the (basic) Thomassen graph. However, the term Thomassen's graph also includes the whole "family" of graphs authored by Carsten Thomassen, but we will stick to the basic one, from Figure 4.

Thomasen's graph is clearly not regular. The graph itself was created as a result of solving the problem of determining the minimum number of graph nodes, *m*, so that for all numbers $n^{3} \quad m$ there is a graph with *n* nodes that is hypohamiltonian.

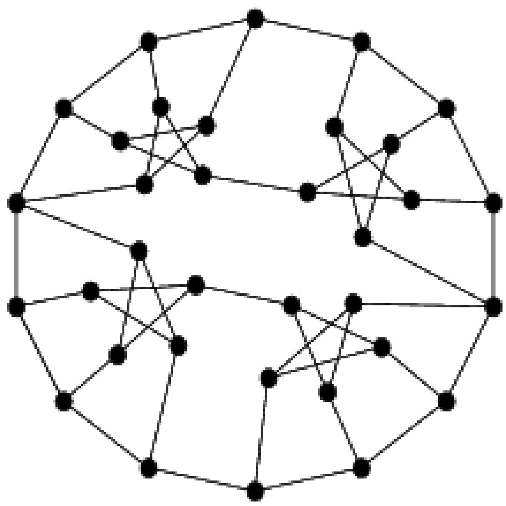

Figure 4: Thomassen graph 34

Thomassen established that there is a hypohamiltonian graph for every number of nodes greater than 42, and also in individual cases, there are hypohamiltonian graphs for node numbers 34, 37, 39 and 40. The smallest of these graphs, for the case of 34 nodes, is exactly the basic Thomassen graph from the bottom Figure 4 (see [6]).

The Thomasen graph is hypohamiltonian. Let's recall the definition of hypohamiltonian graphs:

Definition 2: *Graph G is hypohamiltonian if it is not Hamiltonian, but graph G is Hamiltonian for each node v from the set of nodes V of the graph G.*

So, based on this definition, our graph does not have a Hamiltonian path. This is easily and effectively demonstrated by finding appropriate paths, and given that the representation of the graph in Figure 4 is centrally symmetric in relation to the center of the image, as it is isomorphic with all Thomassen's graphs 34, it is sufficient to check only a few cases instead of all 42.

The next property that often needs to be examined is the chromatic number of the graph and its chromatic polynomial. Mathematical methods (but for many graphs often also an ad-hoc method, direct counting) show that the chromatic number of Thomassen's graph 34 is equal to 3, which is shown in Figure 5.

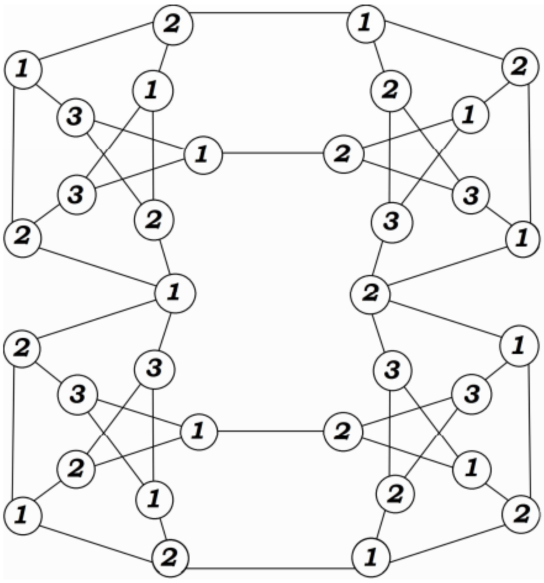

Figure 5: Example of regular 3-coloring of Thomassen's graph 34

Here, we will not dwell on the complete presentation of the procedure for obtaining the chromatic polynomial for the Thomassen graph, but if someone decides to specifically determine the polynomial (which we do not recommend manually), he can do it in the following way:

- The graph is first "reduced" to "smaller" graphs or to some already known cases;

- As a next step, we use two recursive formulas as a consequence of a theorem that gives us the connection of the chromatic polynomial of the graph G and the chromatic polynomials of the two graphs obtained from it:

Theorem 3: *Let G be a simple graph and e its arbitrary branch,* $e \in E(G)$. *Then the following equality holds:*

$$\forall e \in E(G) \quad P(G,t) = P(G-e,t) + P(G \cdot e,t)$$

where G – e denotes the graph obtained from G if we remove its branch e, and $G \cdot e$ *denotes the graph obtained from G if we connect the ends of the branch e into one node.*

This theorem can be applied directly, as its saying reads, but its reverse is also valid:

$$\forall e \in E(G) \quad P(G-e,t) = P(G,t) + P(G \cdot e,t)$$

It turns out that it is simpler to use the direct theorem 3 if the graph *G* has fewer branches, and if there are more, then it is more useful to use the consequence, because then the graph *G* is gradually reduced to a complete graph.

The next thing to do is, using the above formulas, to represent the chromatic polynomial of the graph *G* as a linear combination of the chromatic polynomials of some graphs whose chromatic polynomials have already been determined. We will give some more famous cases:

1. cyclic graphs (Figure 6.1) have chromatic polynomials given by

$$P(G,t) = (-1)^n (t-1) + (t-1)^n$$

2. wheel graphs (Figure 6.2) have chromatic polynomials given by

$$P(G,t) = t\cdot\left[t-2+\left(3-3t+t^2\right)^n\right]$$

3. star graphs (Figure 6.3) have chromatic polynomials given by

$$P(G,t) = t\cdot(t-1)^{n-1}$$

4. "Möbius ladder" graphs (Figure 6.4) have chromatic polynomials given by

$$P(G,t) = -1 + (1-t)^n - (3-t)^n + (-(1-t)^n + (3-t)^n)\cdot t + (3+(-3+t)\cdot t)^n$$

Chromatic polynomials have been found for many known graphs, over 30 of them. At the end of this chapter, we will give a chromatic polynomial for the Thomassen graph 34:

$$\begin{aligned}P(G,t) = {} & (\lambda-1)^2(\lambda-1)\lambda(\lambda^{30} - 47\lambda^{29} + 1083\lambda^{28} - 16305\lambda^{27} + 180324\lambda^{26} - 1561400\lambda^{25} + \\ & + 11015920\lambda^{24} - 65061748\lambda^{23} + 327965806\lambda^{22} - 1431322674\lambda^{21} + 5466847050\lambda^{20} - \\ & - 18424496508\lambda^{19} + 55135240158\lambda^{18} - 147188153740\lambda^{17} + 351719023440\lambda^{16} - \\ & - 754010752102\lambda^{15} + 1451954229559\lambda^{14} - 2512089164775\lambda^{13} + 3902256014515\lambda^{12} - \\ & - 5433059169495\lambda^{11} + 6760525772166\lambda^{10} - 7487041466528\lambda^{9} + 7336866014873\lambda^{8} - \\ & - 6311393434605\lambda^{7} + 4714008483383\lambda^{6} - 3010128105787\lambda^{5} + 1606390745371\lambda^{4} - \\ & - 691675288049\lambda^{3} + 226495327607\lambda^{2} - 50372340381\lambda + 5725152504).\end{aligned}$$

(11)

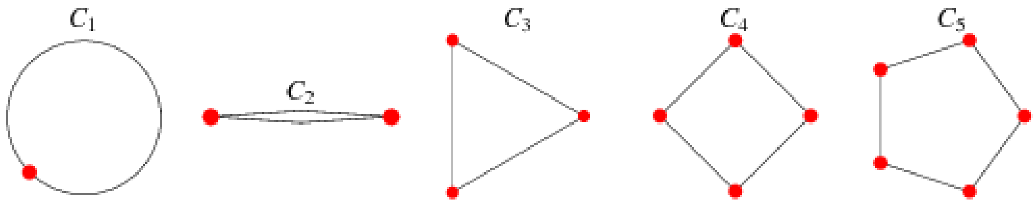

Figure 6.1: The first five cyclic graphs

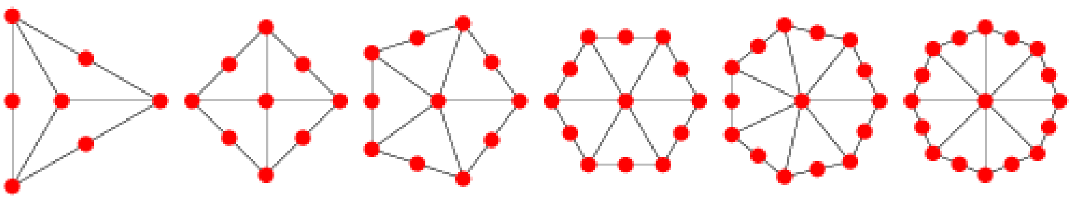

Figure 6.2: The first six wheel graphs

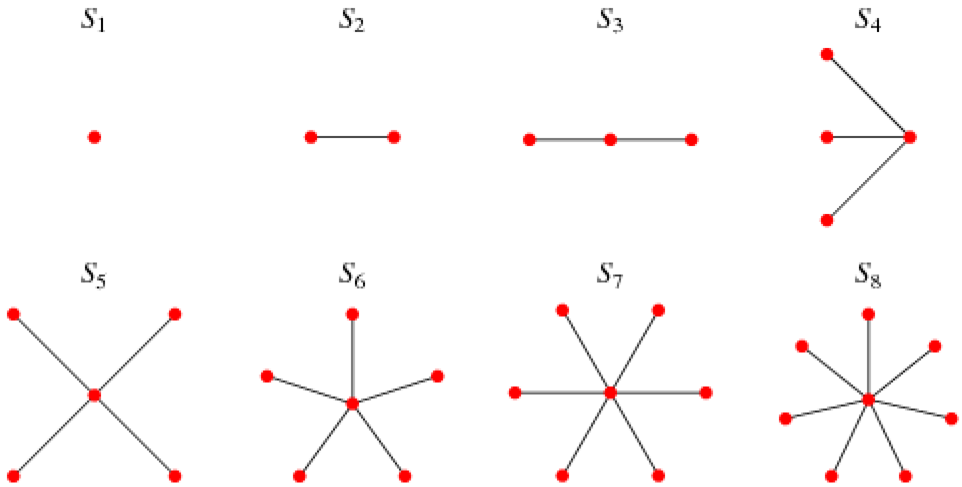

Figure 6.3: The first eight star graphs

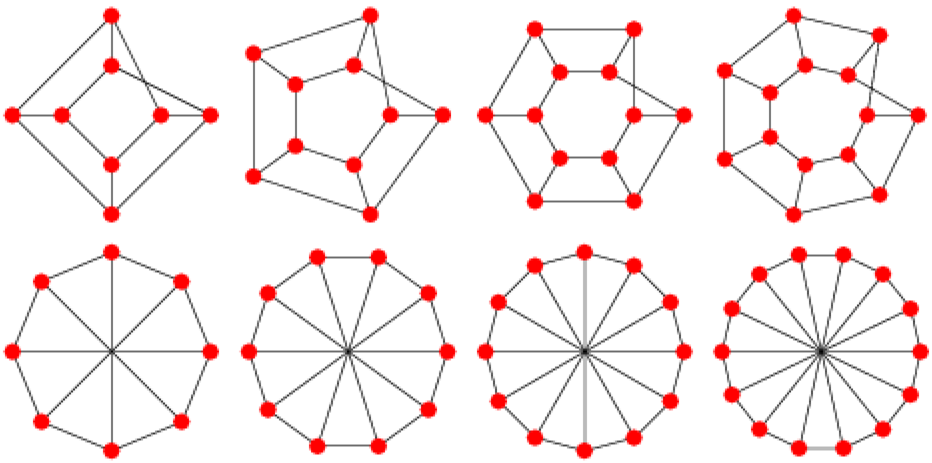

Figure 6.4: The first four „ Möbius ladder " type graphs and their isomorphic images

4. Herschel graph

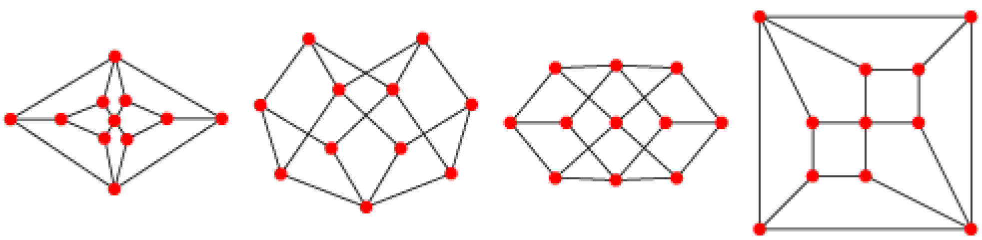

Figure 7: The Herschel graph with some of its isomorphic images

A Herschel graph is a graphical representation used in statistics to visualize the relationship between variables. It is usually used to display data in the form of points, where each point represents a combination of the values of two variables.

To create a Herschel graph, you need:

1. **Basic axes:** The horizontal axis usually shows the independent variable, while the vertical axis shows the dependent variable.
2. **Points:** Each point on the graph represents an observed value for a certain combination of independent and dependent variable.
3. **Trend line**: In some cases, the graph may include a trend line that visually shows the general shape of the relationship between the variables.

The Herschel graph is a polyhedral graph. It is a special type of graph that is used as a counter-example for certain mathematical theorems.

4.1. Basic characteristics of the Herschel graph:

1. **Number of nodes**: Herschel graph has 11 nodes.
2. **Number of branches**: There are 18 branches.
3. **Without triangles**: Herschel's graph is a **triangle-free graph**, which means that there is no subgraph in the form of a triangle (ternary cycle).
4. **Bipartiteness**: Herschel's graph is not bipartite. This means that it cannot be colored with only two colors so that no branch connects nodes of the same color.
5. **Non-planar**: The Herschel graph is not planar, meaning that it cannot be drawn on a plane without intersecting branches.
6. **Counter-example for Wiesing's theorem**: This graph is used as a counter-example for certain mathematical theorems, especially in connection with the coloring of branches and nodes.

4.2. Herschel graph structure:

A Herschel graph can be described as a union of two hexagons (a 6-sided polygon), where two points are connected by a "diagonal" between the polygons, thus providing additional branches.

Usefulness and applications:

1. **Graph theory:** The Herschel graph is used in research related to planarity, coloring and structure of graphs.
2. **Combinatorics:** Also used in combinatorial problems and analysis of algorithms dealing with graph coloring.
3. **Education:** As a counter-example in graph theory, it is used to illustrate how certain theorems do not hold for all graphs.

The history of the creation of Herschel's graph (Figure 7) is closely related to a mathematical game (known as Hamilton's game): it is necessary to find Hamilton's contour in the dodecahedron, but in such a way that the initial and final nodes of that path are the same. For a long time, this game was made in the form of flat cardboard or wooden tiles on which a network of dodecahedrons was drawn, and the nodes of the graph themselves were represented by holes or balls. British astronomer Alexander Stuart Herschel tackled a different approach to this game: he found a graph describing the smallest convex polyhedron that does not have a Hamiltonian path whose start and end nodes are the same.

Of all the established properties of the Hamiltonian graph, perhaps the most significant are its planarity and the property that, if we remove any two nodes from it, we are left with a connected subgraph (this is called 3-connectedness of the graph). These two properties, combined together, can be used in the Steinitz theorem:

Theorem 4: *If the graph G is planar and 3-connected, then it represents the mesh of some convex polyhedron.*

In our case, the Herschel graph represents a network of enneahedron (nonahedron), which is a polyhedron with 9 planar sides (Figure 8). That polyhedron is also the solution of the initial, modified Hamiltonian game considered by Herschel.

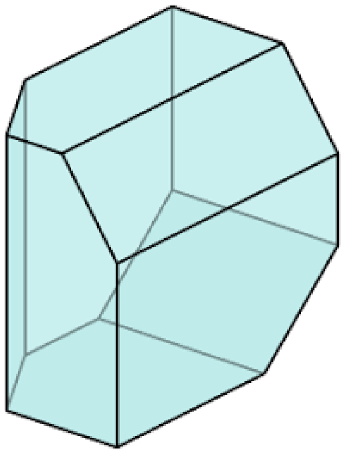

Figure 8: One enneahedron (nonahedron)

The characteristic polynomial of the Herschel graph is

$$P_G(\lambda) = -\lambda^3(\lambda^2 - 11)(\lambda^2 - 3)(\lambda^2 - 2)^2$$

so the following numbers are in its spectrum: 0 (3 times), $\sqrt{2}$ (2 times), $-\sqrt{2}$ (2 times), $\sqrt{3}$ (1 time), $-\sqrt{3}$ (1 time), $\sqrt{11}$ (1 time) and $-\sqrt{11}$ (1 time), or more often in the next record:

$$\sigma(G) = \left\{0^3, \sqrt{2}^2, -\sqrt{2}^2, \sqrt{3}^1, -\sqrt{3}^1, \sqrt{11}^1, -\sqrt{11}^1\right\}$$

Figure 9 shows the bipartition of the Herschel graph: all its nodes are divided into two partitions, so that the nodes marked with H belong to the first partition, and those marked with Y belong to the second partition. Therefore, each branch of the graph has one end in one and the other end in another partition - the branches "alternate".

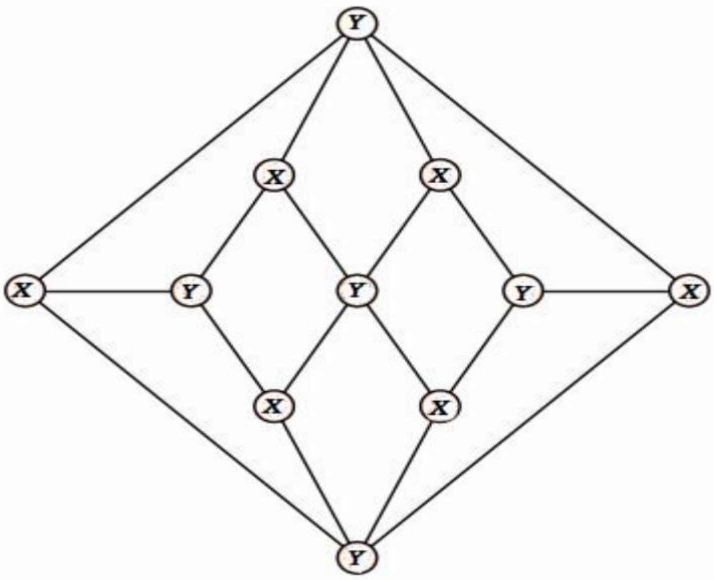

Figure 9: One bipartition of Herschel's graph

Based on the well-known graph Hamiltonianity theorem, which states that a bipartite graph is not Hamiltonian if it has an odd number of nodes, we easily obtain that the Herschel graph is not Hamiltonian, which means that its nodes cannot be visited in a single connected path that would pass through each node exactly once.

The chromatic number of the Herschel graph is also easy to determine by direct inspection. We see, also in Figure 9, that the chromatic number of this graph is equal to 2 if our colors are X and Y. However, sometimes it is easier to apply the following useful theorem:

Theorem 5: *A finite, connected, unoriented graph, without loops and with at least two nodes, whose index is equal to r, is bichromatic if and only if its spectrum also contains the number –r.*

Since the index of the Herschel graph is $\sqrt{11}$, and the spectrum contains and - $\sqrt{11}$, it is a bichromatic graph, without engaging in coloring itself.

In addition to the standard methods for manual determination of properties and corresponding graph polynomials, nowadays it is unthinkable not to use automated programs that do this job in mathematics. Therefore, finding the chromatic polynomial can be left to a program. Using Matlab, we get that the chromatic polynomial of the Herschel graph is

$$P(G,\lambda)=\pi(\lambda)=\lambda(\lambda-1)(\lambda^9-17\lambda^8+136\lambda^7-671\lambda^6+2254\lambda^5-5355\lambda^4+9002\lambda^3-$$
$$-10306\lambda^2+7254\lambda-2371)$$

4.3. It is known that graphs can be examined for hundreds of seemingly independent properties. While in continuous mathematics the derivative and integral, based on the concept of limit values, solve almost all problems, in discrete mathematics, which also includes graph theory, there is nowhere near such a powerful tool (the question is whether it can exist). It seems to the author that from such a large number of properties of graphs by which they are examined, those could be selected that would finally give the correct necessary and sufficient condition to determine when a graph has a Hamiltonian contour. There is also the problem of isomorphism of two graphs, which is also waiting for its solution: when are two graphs isomorphic?

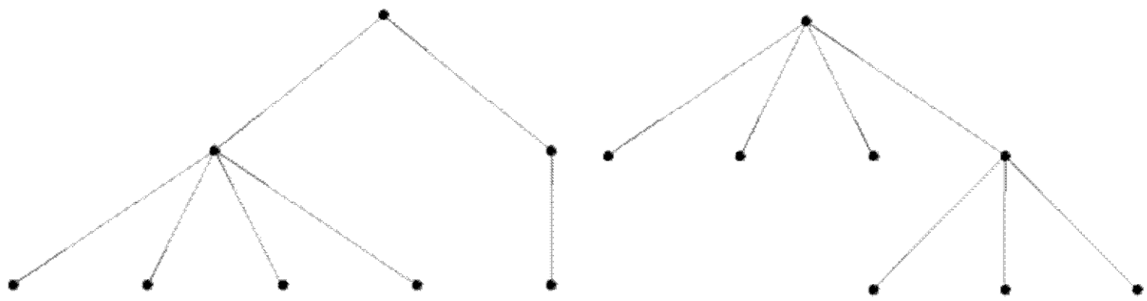

Figure 10: Two non-isomorphic graphs with equal characteristic polynomials

The hunch is that by correctly choosing from the multitude of properties of the graph, the Hamiltonianity problem and the isomorphism problem, as well as many other unsolved problems, can be solved.

For the very end of this paper, let's show with an example how difficult is the analysis of graph isomorphism problems: isomorphic graphs have the same characteristic polynomials. However, the reverse is not valid: let's look at the example of graphs from Figure 10, which are not isomorphic but still have the same characteristic polynomial

$$P_l(G) = l^8 - 7l^6 + 9l^4$$

Also, the graphs from Figure 11 also have the same characteristic polynomials

$$P_l(G) = l^7 - 11l^5 - 10l^4 + 16l^3 + 16l^2$$

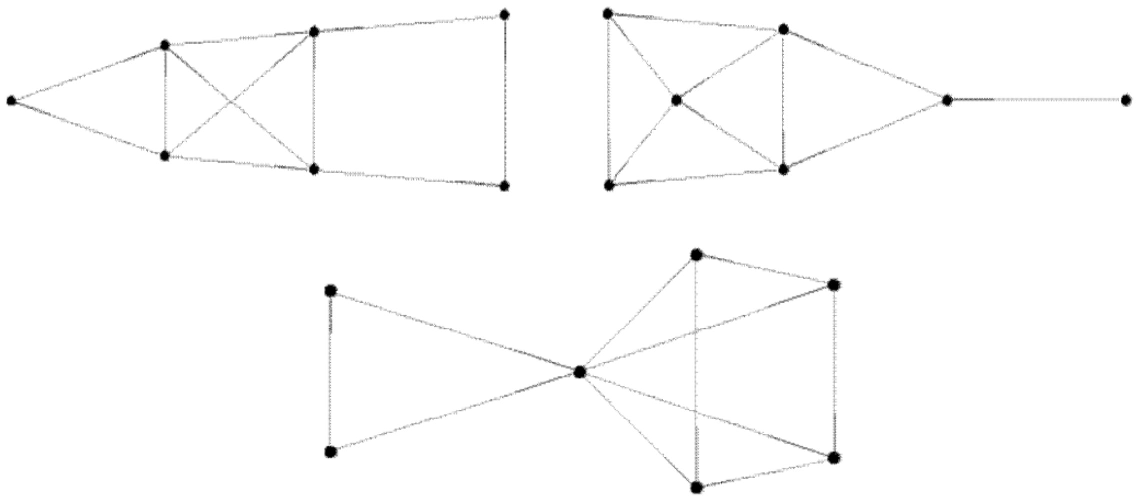

Figure 11: Another example of non-isomorphic graphs with equal characteristic polynomials

5. Conclusion

In writing this paper, we considered that its task should be twofold: the first is to compare three known graphs according to certain characteristics, and the second is to point out the difficulty of solving certain problems that is partially, but only seemingly, removed by the use of computers. Let's just look at an example of the chromatic polynomial of Thomasen's graph 34.

In this paper, the fundamental properties of Grötzsch's, Thomassen's and Herschel's graphs are analyzed and compared in detail from the point of view of spectral and algebraic graph theory. The research confirmed the unique properties of each of these graphs, with Grötzsch's graph being confirmed as the smallest triangle-free graph with chromatic number 4, while Thomassen's and Herschel's graphs showed interesting spectral features that set them apart from other known graphs. By using the eigenvalues and eigenvectors of the Laplacian matrices, a detailed insight into the spectral properties of these graphs was obtained, which enabled a deeper understanding of their structure and potential application in the field of combinatorial optimization and network theory. The analysis also showed the importance of algebraic approaches and the application of modern computer programs in the investigation of graphs with a large number of nodes, such as Thomassen's graph with 42 nodes. It was concluded that the combination of spectral and algebraic theory is necessary for a complete understanding of complex graph structures, which may be of importance for future research in mathematics and applied sciences. Further work in this direction could include the application of these methods to other complex graphs, as well as the investigation of their connection with various problems in graph theory.

Spectral and algebraic characteristics of Grötzsch's, Thomassen's and Herschel's graphs were systematically analyzed and compared. The Grötzsch graph, as the smallest representative of Mycielski graphs with chromatic number 4 without triangles, confirms its role in graph theory as a unique model for studying the chromatic properties of cycle-free graphs of length 3. Detailed analysis shows that this graph retains its efficiency and stability in various applications, especially in colorability research.

The Thomassen graph, with 42 nodes, exhibits specific spectral properties that are significant in the context of problems related to network structures and big data analysis. Calculating the eigenvalues and eigenvectors of the Laplacian of this graph enables a better understanding of its structure, as well as the identification of modalities and roles of individual nodes in the graph. Its complexity and algebraic structure make it a challenging but also useful model for studying optimization in complex network systems.

The Herschel graph, although simpler in terms of the number of nodes and branches, serves as an important counter-example in graph theory. Its non-planar structure and the impossibility of two-coloring emphasize its specificity, while spectral analysis reveals unexpected patterns in the distribution of eigenvalues. These properties make the Herschel graph significant in the study of relations between the structural properties of graphs and their spectral characteristics.

Through a combination of spectral and algebraic analysis, this paper shows that the mentioned graphs are suitable not only for theoretical research, but also for applications in areas such as network optimization, code theory, and big data analysis. It is especially important to emphasize the role of computer tools in the analysis of such complex graphs, where the use of modern software solutions enabled more detailed research and visualization, which would be extremely difficult to achieve with classical methods. Based on these results, further analysis of these and similar graphs is recommended, as well as research into their potential applications in other fields, such as communication theory, biological networks, and information theory. This work also opens the possibility for the development of new algorithms and heuristics that could effectively solve problems related to colorability, optimization, and spectral characteristics of large-dimensional graphs.

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